Int J Game Theory (2007) 35:267–286
DOI 10.1007/s00182-006-0056-z

ORIGINAL PAPER

A spatial voting model where proportional rule leads to two-party equilibria

Francesco De Sinopoli · Giovanna Iannantuoni

Revised: 12 May 2006 / Published online: 3 November 2006

Abstract In this paper we show that in a simple spatial model where the government is chosen under strict proportional rule, if the outcome function is a linear combination of parties' positions, with coefficient equal to their shares of votes, essentially only a two-party equilibrium exists. The two parties taking a positive number of votes are the two extremist ones. Applications of this result include an extension of the well-known Alesina and Rosenthal model of divided government as well as a modified version of Besley and Coate's model of representative democracy.

Keywords Voting · Proportional rule · Nash equilibria

JEL Classification C72 · D72

1 Introduction

In this paper we focus on the strategic behavior of individual voters when elections are called under proportional rule. We embed such an analysis in a very simple spatial model. The spatial voting literature mostly studies plurality elections, while a strategic theory of proportional representation systems still remains an open question, even though there is an increasing interest

F. De Sinopoli (✉) · G. Iannantuoni
Department of Economics, University Carlos III de Madrid,
126 Calle de Madrid, 28903 Getafe, Madrid, Spain
e-mail: fsinopol@eco.uc3m.es

G. Iannantuoni
e-mail: giannant@eco.uc3m.es

in this direction (see, among others, Baron and Diermeier 2001). The main explanation of such a dearth in the literature is in the understanding of how votes are translated into policy, i.e. the policy outcome function. As a matter of fact, while the outcome of a plurality election is unambiguously given by the winner's preferred policy, when we study proportional systems things are more controversial. Typically, voters determine parties' shares of votes in parliament, and elected representatives choose the policy through a government formation process. In this paper we deliberately turn our attention only to the strategic voting stage. To be more clear, we provide a simple but general framework to investigate the strategic behavior of individual voters, who face an outcome function defined as the linear combination of parties' positions weighted with the share of votes that each party gets in the election. The motivation of such a compromise function is to capture the spirit of proportional representation, i.e., any party in parliament is represented in the political process of policy determination, with a weight that is proportional to its share of votes.[1] Furthermore, such an outcome is the utilitarian solution of a bargaining process among politicians with a quadratic loss function and, then, the result of a proto-coalition bargaining procedure, as modelled in Baron and Diermeier (2001) when the status quo is quite negative for the elected politicians (see Remark 1, page 9).

Given such a compromise function, we build a game in which a finite number of voters may cast their ballot for any of a finite set of parties. We prove that essentially only a two-party equilibrium exists, in which voters vote only for the two extremist parties. We want to raise from now the two main strengths of this paper: solving the game with a finite number of players, and not restricting the number of parties on the political scene.

A key message of this paper is the analysis of a game with a continuum of voters as the limit game of games with a finite number of voters. Obviously, assuming a continuum of voters implies that each player is negligible to the outcome. For this reason we believe that assuming a finite number of players is crucial to study the strategic behavior of voters. Hence, we start by analyzing games with a finite number of voters, fully capturing the strategic incentive of the voters to vote for the extremists. More precisely, we prove that, with a finite number of voters, essentially a unique Nash equilibrium exists, characterized by an outcome (a cutpoint) such that any voter to its right votes for the rightmost party and any voter to its left votes for the leftmost party. The intuition of the result is obvious: strategic voters misrepresent their preferences by voting for the extremist parties in order to drag the policy outcome toward their bliss policy.[2] The result above allows us to analyze a game with a continuum of voters as the limit game of games with a finite number of voters. In such a case

[1] For a similar policy outcome see, for example, Alesina and Rosenthal (2000) and Ortuño-Ortin (1997).

[2] The incentive to vote for an extreme is given by the maximal effect that such a vote has on the outcome. Vice versa, if the policy outcome is the median, voting sincerely is a dominant strategy because the effect of each vote, but the median one, is solely "directional" (i.e., any vote to the left of the median has the same effect as well as any vote to the right).

each voter behaves as if he were decisive, and the "equilibrium" outcome is the policy obtained with every voter to its left voting for the leftmost party and every voter to its right for the rightmost party.

The second important point we mentioned above is that by not restricting the number of parties on the scene, we're able to handle one of the most analyzed question in the voting literature, that is if and how electoral rules affect the formation and the survival of political parties in mass elections. Duverger (1954) first observed a tendency to have just two serious candidates in plurality rule elections, whereas proportional systems are more likely to have several parties. Riker (1982) in a famous paper precisely defined *Duverger's Law* and *Duverger's Hypothesis* . *Duverger's Law* states that "the simple-majority single-ballot system (i.e. simple plurality rule) favors the two-party system" (Duverger 1954:217). *Duverger's Hypothesis* states that "proportional representation favors multipartyism" (Duverger 1954:239). *Duverger's Law* and *Hypothesis* have established themselves as two of the premier empirical regularities in political science. The most common explanation of *Duverger's Law* relies on the role that strategic voting may have in plurality rule elections.[3] Duverger (1954:226) explained: "in cases where there are three parties operating under the simple majority single-ballot system the electors soon realize that their votes are wasted if they continue to give them to the third party: hence their natural tendency to transfer their vote to the less evil of its two adversaries in order to prevent the success of the greater evil" . This explanation given by Duverger has been translated into strategic voting by formal models (see Palfrey 1989; Cox 1994; Myerson and Weber 1993; Fey 1997). Few scholars focused on *Duverger's Hypothesis*, in general assuming that strategic voting is absent under proportional representation, hence explaining multipartyism [4] (see Cox (1997) for a lucid discussion on this point). This paper clearly points out that strategic voting may have devastating effect also under proportional representation, with the effect of considerably reducing the number of parties for which voters vote.

Another contribution of this paper is to apply the general result to well-known voting models. First, we discuss the multi-party version of a divided government model, where in the spirit of Alesina and Rosenthal's (1996) analysis, we obtain a *moderation* result: we show that the more rightist the president is, more votes are taken by the leftist party. We then present an example with three parties, in which the two extremist parties take votes in the legislative election, while the party located at center wins the executive election. An interesting implication is that more complex institutional systems may have more political parties than their components would have separately. Second, we study, following Besley and Coate (1997), endogenous candidacy, finding that, when the cost of candidacy is small, only the two extremists will become candidates.

---

[3] We like to cite Riker's words (1982:764): "The evidence renders it undeniable that a large amount of sophisticated voting occurs - mostly to the disadvantage of the third parties nationally so that the force of Duverger's psychological factor must be considerable".

[4] Leys (1959) and Sartori (1968) were the first scholars to claim that strategic voting is also effective under proportional representation with the consequence to reduce the number of parties.

Before introducing the model, we present some stylized facts that suggest this is of more than theoretical interest.

Proportional representation and a two-party system. Riker himself, after the analysis of four counterexamples (Australia, Austria, Germany, and Ireland) to *Duverger's Hypothesis*, concluded that "we can therefore abandon Duverger's Hypothesis in its deterministic form" (1982:760).[5] Two cases seem particularly meaningful.

The first case is Austria, defined by Riker as a "true counterexample" (1982: 758) that experienced a stable two-party system under proportional representation.[6] The two major parties, the Christian Socialist (OVP) and the Social Democrats (SPO), were essentially duopolists with eighty to ninety per cent (or more) of the votes from 1945 to 1987 (see Engelmann 1988:87).

The second case is Ireland, considered by Riker (1982:758) as "a devastating counterexample" to *Duverger's Hypothesis*. The reason is that proportional representation [7] favored a decrease in the number of parties: since the elections of 1927, when there were seven parties and fourteen independents, the number of parties decreased, and from the election of 1969 three parties were on the scene together with a few independent parties. From the elections of 1972 a stable "two-party and half" system (Carty 1988:224) was founded.

Moderation in the multiparty version of Alesina and Rosenthal's model. We define a two-stage game where first there is an election of the president with plurality rule, and then an election of a legislature with proportional rule. The main finding is that the share of votes taken by the leftmost party in the legislative election is increasing in the position of the president, i.e., the more rightist is the president, the more votes will be taken by the leftmost party. This theoretical prediction can explain cases when, in presidential countries, midterm elections are dominated by the parties losers in the presidential race (see Shugart 1995 for a insightful discussion about this point,[8] and Alesina and Rosenthal 1995).

The paper is organized as follows. In Sects. 2 and 3 we present the general model and our main results. In Sect. 4 we present two applications, one to the Alesina and Rosenthal (1996) model of divided government in Subsect. 4.1, and one to the Besley and Coate (1997) model in Subsect. 4.2. Finally, Sect. 5 concludes the paper.

[5] Cox (1997) offers a exhaustive analysis on the relation between electoral rules and the resulting number of parties.

[6] More precisely, the electoral system is as follows. Each list receives as many seats as its vote contains full Hare quotas, and those seats are then allocated to the list's candidates, in accordance with the list order. Seats unallocated in the first step are aggregated in accordance with each secondary list's vote and then reallocated to the list's candidate (see Cox 1997).

[7] The Irish system is proportional representation by means of the single transferable vote (STV). Under STV the voter has the opportunity to indicate a range of preferences by placing numbers in correspondence with candidates' names on the ballot paper. A vote can be transferred from one candidate to another if it is not required by the prior choice to make up that candidate's quota (or if, as a result of poor support, that candidate is eliminated from the contest).

[8] Shugart considers eleven countries, including France, Chile, and El Salvador.

2 The basic model

We define the simplest framework to analyze an election called with proportional rule:

Policy Space. The policy space $\mathbb{X}$ is a closed interval of the real line, and without loss of generality we assume $\mathbb{X} = [0,1]$.

Parties. Parties are fixed both in number and in their positions,[9] in that there is no strategic role for them: there is an exogenously given set of parties $M = \{1, \ldots, k, \ldots m\}$ $(m \geq 2)$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0,1]$.

Strategy. Given the set of parties M, each voter can cast his vote for a party.[10] The pure strategy space of each player i is $S_i = \{1, \ldots, k, \ldots, m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a one in position k, which represents the vote for party k.[11]

A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, \ldots \sigma_i^k, \ldots, \sigma_i^m)$ where each σ_i^k represents the probability that player i votes for party k.

Policy outcome. The position of the government, i.e., the policy outcome, is a linear combination of parties' policies, each coefficient being equal to the corresponding share of votes. Given a pure strategy combination $s = (s_1, s_2, \ldots, s_n)$, $v(s) = \sum_{i \in N} \frac{s_i}{n}$ is the vector representing for each party its share of votes. Hence the policy outcome can be written as:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s). \tag{1}$$

Voters. Each voter i is characterized by his bliss point $\theta_i \in \Theta = [0,1]$. Voters' preferences are single peaked. We stress that this is the only assumption needed to reach the result for pure strategy equilibria. To analyze mixed strategy equilibria, we assume that a fundamental utility function, which represents preferences, $u : \Re^2 \to \Re$ exists and is continuously differentiable with respect to the first argument,[12] that is, $u_i(X) = u(X, \theta_i)$.

Given the set of parties and the utility function u, a finite game Γ is characterized by a set of players $N = \{1, \ldots, i, \ldots, n\}$ and their bliss points. Given $\Gamma = \left\{N, \{\theta_i\}_{i \in N}\right\}$ we denote by $H^{\Gamma}(\theta)$ the distribution of players' bliss points,[13] i.e., $H^{\Gamma}(\theta^*)$ is the proportion of players with a bliss point less than or equal to θ^*.

---

[9] We will relax these assumptions in the application to Besley and Coate's model of represenative democracy (1997), when parties will be endogenous.

[10] In this paper we do not allow for abstention. We cannot claim that this assumption is neutral. In our proof we use the fact that, as the number of players goes to infinity, the weight of each player goes to zero, and this result does not hold if a large number of voters abstain.

[11] The description of a ballot as a vector is borrowed from Myerson and Weber (1993). In order to simplify the notation the same symbol denotes a party and the strategy of voting for it.

[12] Hence, by single-peakedness, $\forall \bar{x}_2 \in [0,1]$, $\frac{\partial u(x_1, \bar{x}_2)}{\partial x_1} \gtreqless 0$ for $x_1 \lesseqgtr \bar{x}_2$ and $x_1 \in [0,1]$.

[13] Sometimes we will identify a player with his bliss point.

The utility that player i gets under the strategy combination s is:

$$U_i(s) = u(X(s), \theta_i).$$

Given a mixed strategy combination $\sigma = (\sigma_1, \ldots, \sigma_n)$, because players make their choice independently of each other, the probability that $s = (s_1, s_2, \ldots, s_n)$ occurs is:

$$\sigma(s) = \prod_{i \in N} \sigma_i^{s_i}.$$

The expected utility that player i gets under the mixed strategy combination σ is:

$$U_i(\sigma) = \sum \sigma(s) U_i(s).$$

In the following, as usual, we shall write $\sigma = (\sigma_{-i}, \sigma_i)$, where $\sigma_{-i} = (\sigma_1, \ldots \sigma_{i-1}, \sigma_{i+1}, \ldots \sigma_n)$ denotes the $(n-1)$ −tuple of strategies of the players other than i. Furthermore, s_i will denote the mixed strategy σ_i that gives probability one to the pure strategy s_i.

3 Nash equilibria

In this section we analyze the equilibria of the game defined above. First, we analyze voters' behavior when only pure strategies are allowed. We show that in any pure strategy Nash equilibrium, voters vote only for the extreme parties, except for a neighborhood of ideal points inversely related to the number of players. We define then the cutpoint outcome, i.e., the only outcome obtained with any voter strictly on its right voting for the rightmost party and any voter strictly on its left voting for the leftmost party. If the cutpoint outcome does not coincide with any voter's bliss point, then the strategy combination defining the cutpoint is a pure strategy Nash equilibrium of the game.

As nothing assures us that this sufficient condition for the existence of a pure strategy equilibrium is satisfied, or that mixed strategy equilibria behave completely analogously, we extend the analysis to the case when voters are allowed to play mixed strategies. We prove the main result of this paper: in any mixed strategy equilibrium any player on the right of the cutpoint outcome votes for the rightmost party, and any player on the left of the cutpoint outcome votes for the leftmost party, except for a neighborhood inversely related to the number of voters.

We then study games with a continuum of voters as limits of games with a finite number of voters, i.e., each voter behaves as if he could be decisive. The previous analysis, developed for games with a finite number of players, allows us to consider the cutpoint outcome as the "right" solution of games with a continuum of voters.

In order to simplify the notation, in the following we will denote L the leftmost party and R the rightmost (i.e., $L = \arg\min_{k \in M} \zeta_k$, $R = \arg\max_{k \in M} \zeta_k$).[14]

3.1 Pure strategy equilibria

We start by analyzing the pure strategy equilibria in order to stress the intuition behind the result, that is, strategic voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their bliss policy. First, we stress that only the assumption of single peakedness of voters' preferences is needed to get the result. We prove that every pure strategy equilibrium is such that (except for a neighborhood whose length is inversely proportional to the number of players) everybody votes for one of the two extremist parties.

Proposition 1 *Let s be a pure strategy equilibrium of a game Γ with n voters*:

(α) *if* $\theta_i \leq X(s) - \frac{1}{n}(\zeta_\mathrm{R} - \zeta_\mathrm{L})$ *then* $s_i = \mathrm{L}$,
(β) *if* $\theta_i \geq X(s) + \frac{1}{n}(\zeta_\mathrm{R} - \zeta_\mathrm{L})$ *then* s_i=R.

Proof (α) Notice that if $X(s_{-i}, \mathrm{L}) \geq \theta_i$ then, by single-peakedness, L is the only best reply, for player i, to s_{-i} (i.e., $\forall k \neq L$, $X(s_{-i}, k) > X(s_{-i}, \mathrm{L})$). Because $X(s_{-i}, \mathrm{L}) = X(s) - \frac{1}{n}(\zeta_{s_i} - \zeta_\mathrm{L}) \geq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, the assumption $\theta_i \leq X(s) - \frac{1}{n}(\zeta_\mathrm{R} - \zeta_\mathrm{L})$ implies that L is the unique best reply, for player i, to s_{-i}. (β) A symmetric argument holds. □

The proposition above implies that in every pure strategy Nash equilibrium of a game, the proportion of votes taken by the less extreme parties goes to zero as n goes to infinity.[15]

At this point, it is natural to give the following definition.

Given a game Γ and its distribution of bliss points $H^\Gamma(\theta)$, let $\tilde{\theta}^\Gamma$, defined as cutpoint policy, be the unique policy outcome obtained with voters strictly on its left voting for L and voters strictly on its right voting for R, i.e., let $\tilde{\theta}^\Gamma$ be implicitly defined by:

$$\tilde{\theta}^\Gamma \in \zeta_\mathrm{L} \bar{H}^\Gamma(\tilde{\theta}^\Gamma) + \zeta_\mathrm{R}(1 - \bar{H}^\Gamma(\tilde{\theta}^\Gamma))$$

where $\bar{H}^\Gamma$ is the correspondence defined by $\bar{H}^\Gamma(\theta) = \left[\lim_{y \to \theta^-} H^\Gamma(y), H^\Gamma(\theta)\right]$.

Let us assume that no player's preferred policy coincides with the cutpoint outcome. The strategy combination given by any voter strictly on the left of $\tilde{\theta}^\Gamma$ voting for the leftmost party, and any voter strictly on the right of $\tilde{\theta}^\Gamma$ voting for the rightmost party is a pure strategy Nash equilibrium. As a matter of fact, no

[14] We assume that there is only one party at ζ_L as well as at ζ_R. This assumption simplifies the notation, but it does not affect the result. Without this assumption, if we denote L and R the set of extremist parties, everything still holds.

[15] At least if voters' bliss points are sufficiently spread out.

player on the left of the cutpoint outcome has an incentive to vote for any party different from L, because doing so would push the policy outcome further away from his preferred policy. The same argument holds for any player on the right of the policy outcome. We can, then, state the following proposition:

Proposition 2 *If* $\theta_i \neq \tilde{\theta}^{\Gamma}$ $\forall i \in N$, *then the strategy combination given by* $\forall \theta_i < \tilde{\theta}^{\Gamma}$ $s_i = \mathrm{L}$ *and* $\forall \theta_i > \tilde{\theta}^{\Gamma}$ $s_i = \mathrm{R}$ *is a pure strategy Nash equilibrium of the game* Γ.

It is clear that nothing assures us that pure strategy equilibria exist; moreover we have to check if mixed strategy equilibria prescribe a dramatically different behavior for individual voters.

3.2 Mixed strategy equilibria

We analyze the case when players are allowed to play mixed strategies. In order to pursue this analysis we have to assume also that the utility function u is continuously differentiable with respect to the first argument.[16]

We recall that, given the set of parties M and the utility function u, a game Γ is characterized by the set of players and their bliss points. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\bar{\mu}^{\sigma} = \sum_{i \in N} \frac{\sigma_i}{n}$. With abuse of notation, let $X(\bar{\mu}^{\sigma}) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^{\sigma}$.

We can state the following proposition:

Proposition 3 $\forall \varepsilon > 0, \exists n_0$ *such that* $\forall n \geq n_0$ *if* σ *is a Nash equilibrium of a game* Γ *with* n *voters*, *then*:

(α) *if* $\theta_i \leq X(\bar{\mu}^{\sigma}) - \varepsilon$ *then* $\sigma_i = \mathrm{L}$,
(β) *if* $\theta_i \geq X(\bar{\mu}^{\sigma}) + \varepsilon$ *then* $\sigma_i = \mathrm{R}$.

Proof See Appendix. □

In the appendix we will show that $\bar{\mu}^{\sigma}$ is the expected vote shares for the parties. The proposition above says that in any Nash equilibrium, except for a neighborhood whose length decreases as the number of players increases, everybody to the left of $X(\bar{\mu}^{\sigma})$ votes for L, while everybody to its right votes for R. We highlight the effectiveness of the proof, i.e. the n_0 is explicitly calculated as a function of ε, of the number of parties m, and of the shape of the utility function $u(X, \theta)$.

[16] To study mixed strategies equilibria some "cardinal" assumptions on the utility function are needed. Because we use the mean value theorem the assumption we have made is the differentiability one, which seems to be the weakest one to get the results. Furthermore, the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ in X guarantees the existence, for each player, of a lower bound on the number of players for which the results hold. The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ in θ assures that a bound can be found independently of the set of players.

Using the definition of cutpoint policy outcome, we can state our main result: in large electorate essentially a unique Nash equilibrium of the game exists:

Corollary 4 (main result) $\forall \eta > 0$, $\exists n_1$ *such that* $\forall n \geq n_1$ *if* σ *is a Nash equilibrium of a game* Γ *with* n *voters*, *then*:

(α) *if* $\theta_i \leq \tilde{\theta}^{\Gamma} - \eta$ *then* $\sigma_i = \mathrm{L}$,
(β) *if* $\theta_i \geq \tilde{\theta}^{\Gamma} + \eta$ *then* $\sigma_i = \mathrm{R}$.

Proof Fix η and, in Proposition 3, take $\varepsilon = \frac{\eta}{2}$. For the corresponding n_0 it is easy to see that if $n \geq n_0$ and σ is a Nash equilibrium of Γ, $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} \leq X(\bar{\mu}^{\sigma}) \leq \tilde{\theta}^{\Gamma} + \frac{\eta}{2}$. In fact, suppose by contradiction that $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} > X(\bar{\mu}^{\sigma})$ then Proposition 3 implies that all voters to the right of $\tilde{\theta}^{\Gamma}$ vote for the rightmost party and hence $\tilde{\theta}^{\Gamma} \leq X(\bar{\mu}^{\sigma})$, contradicting $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} > X(\bar{\mu}^{\sigma})$. Analogously for the second inequality. Hence $\tilde{\theta}^{\Gamma} - \eta \leq X(\bar{\mu}^{\sigma}) - \frac{\eta}{2}$ and $\tilde{\theta}^{\Gamma} + \eta \geq X(\bar{\mu}^{\sigma}) + \frac{\eta}{2}$, which, with Proposition 3, complete the proof. □

Every equilibrium conforms to such a cutpoint, and hence, for n large enough, only the two extremist parties take a significant amount of votes.

Remark 1 On the outcome function: It is worthwhile to highlight the assumptions on the outcome function needed for the results above stated. First, the outcome function has to be continuous in the share of votes the parties get. Moreover, we need also that $\forall s_{-i}$, $X(s_{-i}, \mathrm{R}) > X(s_{-i}, k) > X(s_{-i}, \mathrm{L})$ $\forall k \neq \mathrm{L}, \mathrm{R}$. Hence, our proofs extend to any continuous and monotonic transformation of (1). From those observations, it is immediate to realize that, when the number of parties is equal to two, the outcome function can be any continuous and strictly increasing function in the share of votes of the rightist party.[17] However, the linearity of the outcome function has a nice justification: it is the utilitarian solution of a bargaining process among politicians with a quadratic loss function. Hence, it is the result of a bargaining process of government formation *á la* Baron and Diermeier (2001), under the assumption that the status quo is quite negative for parliamentary members. This is a weak assumption if the status quo is given by new election where parliamentary members face the risk of not being reelected, and the cost of staying out of the legislature is sufficiently large, as in Austen-Smith and Banks (1988).

3.3 Games with a continuum of voters

We now analyze analogous games with a continuum of voters. In such games every strategy combination is a Nash equilibrium, because each player's vote does not affect the outcome. Nevertheless, the results obtained in the previous pages legitimate the analysis of a game with a continuum of players as the limit game of games with a finite number of players. In such a case each voter behaves

[17] See Alesina and Rosenthal (1996), Grossman and Helpman (1999) and Iannantuoni (2004).

as if he could be decisive, and the "equilibrium" outcome is the policy obtained with every voter to the left of the policy outcome voting for the leftmost party and every voter to the right for the rightmost party.

Let the bliss point distribution function characterizing the game with a continuum of voters $H(\theta)$ be continuous and strictly increasing, and let $\tilde{\theta}$ be the unique policy outcome obtained with voters on the left of $\tilde{\theta}$ voting for L and voters on the right voting for R, i.e., $\tilde{\theta}$ is the unique solution of

$$\tilde{\theta} = \zeta_{\mathrm{L}} H(\tilde{\theta}) + \zeta_{\mathrm{R}}(1 - H(\tilde{\theta})).$$

The previous analysis implies that $\tilde{\theta}$ is the "equilibrium" of the game characterized by $H(\theta)$ when this game is seen as a limit of finite games.[18]

Remark 2 Comparison with the median: As far as the equilibrium policy is concerned, it is quite natural the comparison with the median voter's position. As a matter of fact, the median has a special appeal as the result of compromise in a one-dimensional political space. Let θ_{m} be the median position, it is immediate to verify that, if the positions of the leftmost and the rightmost party are symmetric around $1/2$, if $\theta_{\mathrm{m}} < 1/2$ then $\theta_{\mathrm{m}} < \tilde{\theta} < 1/2$, as well as whenever $1/2 < \theta_{\mathrm{m}}$ then $1/2 < \tilde{\theta} < \theta_{\mathrm{m}}$. This result clearly suggests that the equilibrium policy obtained with proportional representation is more "moderate" than the median outcome. [19]

4 Two applications

In this section we consider two well-known models of political economy.

The first one is the Alesina and Rosenthal (1996) model of divided government. In such a model the policy outcome is described through a compromise between the executive, elected by plurality rule, and the legislature, elected by proportional rule. Considering the two-stage game in which first the president and then the legislature is elected, backward induction implies that in the second stage only the two extremists will obtain votes. More importantly, in the spirit of Alesina and Rosenthal's analysis, we obtain a *moderation* result: we show that further right the president is, the more votes are taken by the leftmost party in the legislative election.

The second model we consider is that of Besley and Coate (1997), where the set of candidates is endogenous. Each citizen decides whether to become a

[18] More precisely, given a sequence of finite games $\{\Gamma_t\}_{t=1}^{\infty}$ we have an associated sequence of distributions of players' bliss points $\{H_t(\theta)\}_{t=1}^{\infty}$. For each game Γ_t, let θ_t^{L} be the rightmost player such that in any equilibrium he and all the players on his left vote for L, and let θ_t^{R} be the leftmost player such that in any equilibrium he and all the players on his right vote for R. By Corollary 4, if the sequence of distributions converge to $H(\theta)$ then both the sequences $\left\{\theta_t^{\mathrm{L}}\right\}_{t=1}^{\infty}$ and $\left\{\theta_t^{\mathrm{R}}\right\}_{t=1}^{\infty}$ converge to $\tilde{\theta}$.

[19] Results in the same flavor are obtained, in different settings, by Austen-Smith (2001), and Morelli (2004).

candidate, incurring a cost, or not. Our result implies that as the cost of candidacy goes to zero, only the two extremist citizens will be candidates.

We analyze both models assuming a continuum of voters and under the assumption that the distribution of bliss points $H(\theta)$ is strictly increasing and continuously differentiable, because in such a case we have uniqueness and differentiability of the "equilibrium".

4.1 Divided government

Alesina and Rosenthal (1996) describe the formation of national policies as the result of institutional complexity captured by the existence of two decisional branches of the government: the executive (i.e., the president), elected under plurality rule, and the legislature, elected under proportional rule. The main implication of this model is that "divided government" can be explained through the behavior of voters with intermediate (that is, situated between parties' announced positions) preferences, who take advantage of the institutional structure to balance the plurality of the winning party in the executive by voting for the opposite party in the legislative election. The main result of Alesina and Rosenthal can be expressed as: *a party receives more votes in the legislative election if it has lost the executive election*.

In this section, we limit the analysis to a two-stage game in which first the president and then the legislature is elected, and we show that analogous results hold for any finite number of parties. More precisely, the results presented in the previous section imply that only the two extremists take votes in the proportional stage. Interestingly, we show that the further to the right the president is, the more votes are taken by the leftmost party. As shown above, our solution rests on *purely individual behavior*, viewing the game with a continuum of players as a limit of finite games. Alesina and Rosenthal's solution is instead based on coalitions, to circumvent the difficulties arising from the fact that with a continuum of voters every vote is negligible to the outcome.

Let us recall the timing of the game: in the first stage players vote, by plurality rule, for the president, and then in the second stage they vote for the legislature, elected with proportional rule.

Given the result of the elections, let the position of the legislature be given by

$$X^{\text{leg}} = \sum_{k=1}^{m} \zeta_k v_k$$

where v_k denotes the share of votes taken by party k, and let the policy outcome be a convex combination of presidential and legislative positions:

$$X = (1-\alpha)\zeta_{\text{P}} + \alpha X^{\text{leg}}$$

where ζ_{P} denotes the position of the party winning the presidential election and $0 < \alpha < 1$.

Solving this game by backward induction, it is evident that, given the election of the president P, the proportional stage is equivalent to the "proportional game" studied in the previous sections, with translated positions of the parties. In other words, given P, we have to analyze the "proportional game" with the set of parties M^{P} where each party k is characterized by the policy

$$\zeta_k^{\mathrm{P}} = (1-\alpha)\zeta_{\mathrm{P}} + \alpha\zeta_k.$$

The results of the previous sections imply that the equilibrium is such that only the two extremist parties[20] L and R take votes. Moreover, the cutpoint strategy $\tilde{\theta}^{\mathrm{P}}$ is given by the unique solution to:

$$\tilde{\theta}^{\mathrm{P}} = \zeta_{\mathrm{L}}^{\mathrm{P}} H(\tilde{\theta}^{\mathrm{P}}) + \zeta_{\mathrm{R}}^{\mathrm{P}}(1 - H(\tilde{\theta}^{\mathrm{P}}),$$

which can be re-written as:

$$\tilde{\theta}^{\mathrm{P}} = (1-\alpha)\zeta_{\mathrm{P}} + \alpha\zeta_{\mathrm{R}} - \alpha(\zeta_{\mathrm{R}} - \zeta_{\mathrm{L}})H(\tilde{\theta}^{\mathrm{P}}). \tag{2}$$

Hence we have

$$\frac{\partial\tilde{\theta}^{\mathrm{P}}}{\partial\zeta_{\mathrm{P}}} = \frac{1-\alpha}{1+\alpha H'(\tilde{\theta}^{\mathrm{P}})(\zeta_{\mathrm{R}} - \zeta_{\mathrm{L}})} > 0. \tag{3}$$

Because $H(\tilde{\theta}^{\mathrm{P}})$ represents the share of votes taken in the legislative election by the leftmost party and $H(\theta)$ is strictly increasing, (3) implies that such a share is increasing in the position of the president. Hence also in multi-party systems, we have a *moderation* result.

The main difficulties in analyzing such a model arise at the presidential stage. It is well known that plurality games in multi-candidate elections display multiplicity of equilibria, and that in order to have sensible solution strong refinements, such as Mertens' stability, seem to be needed (see De Sinopoli 2000, for a discussion on this point).

Nevertheless, for some specification of the parameters of the model, the plurality stage can be solved by iterated elimination of dominated strategies. The following example shows a situation in which the plurality stage is dominance solvable and the center wins the presidential election, while the two extremists win the legislative one. A very interesting implication of this example is that richer institutional system may display more political parties than their components would have separately.

Example There are three parties L,C, and R with $\zeta_{\mathrm{L}} = 0, \zeta_{\mathrm{C}} = \frac{1}{2}$, and $\zeta_{\mathrm{R}} = \frac{3}{5}$. Suppose the voters' bliss points are distributed uniformly on $[0,1]$, with symmetric utility functions and $\alpha = \frac{1}{6}$.

[20] Obviously we have $\mathrm{L} = \arg\min_k \zeta_k = \arg\min_k \zeta_k^{\mathrm{P}}$ and $\mathrm{R} = \arg\max_k \zeta_k = \arg\max_k \zeta_k^{\mathrm{P}}$.

If we solve the game backward, Eq. (2) gives us the equilibrium outcome for each possible president. It is not difficult to compute that

$$\tilde{\theta}^{\mathrm{L}} = \frac{1}{11}, \quad \tilde{\theta}^{\mathrm{C}} = \frac{31}{66}, \quad \text{and} \quad \tilde{\theta}^{\mathrm{R}} = \frac{6}{11}.$$

In the first stage, hence, citizens choose with plurality among $\tilde{\theta}^{\mathrm{L}}, \tilde{\theta}^{\mathrm{C}}$, and $\tilde{\theta}^{\mathrm{R}}$. Obviously we have the following preference orders on the election of L, C, and R as president:

$$\begin{array}{ll}
0 \le \theta_i < \dfrac{37}{132} & \mathrm{L} \succ_i \mathrm{C} \succ_i \mathrm{R} \\[2ex]
\theta_i = \dfrac{37}{132} & \mathrm{L} =_i \mathrm{C} \succ_i \mathrm{R} \\[2ex]
\dfrac{37}{132} < \theta_i < \dfrac{7}{22} & \mathrm{C} \succ_i \mathrm{L} \succ_i \mathrm{R} \\[2ex]
\theta_i = \dfrac{7}{22} & \mathrm{C} \succ_i \mathrm{L} =_i \mathrm{R} \\[2ex]
\dfrac{7}{22} < \theta_i < \dfrac{67}{132} & \mathrm{C} \succ_i \mathrm{R} \succ_i \mathrm{L} \\[2ex]
\theta_i = \dfrac{67}{132} & \mathrm{C} =_i \mathrm{R} \succ_i \mathrm{L} \\[2ex]
\dfrac{67}{132} < \theta_i \le 1 & \mathrm{R} \succ_i \mathrm{C} \succ_i \mathrm{L}
\end{array}$$

In a plurality election, the strategy of voting for the least preferred candidate is dominated (by voting for the most preferred). In the reduced game obtained by eliminating such strategies, the players have the following strategies:

$$\begin{array}{ll}
\theta_i < \dfrac{7}{22} & \mathrm{L}, \mathrm{C} \\[1.5ex]
\theta_i = \dfrac{7}{22} & \mathrm{C} \\[1.5ex]
\theta_i > \dfrac{7}{22} & \mathrm{R}, \mathrm{C}.
\end{array}$$

In this reduced game there is no chance of candidate L being elected president, because he takes at most $\frac{7}{22}$ of the total number of votes. Hence, voting for him is dominated, as are voting for R if $\frac{7}{22} < \theta_i < \frac{67}{132}$ and voting for C if $\frac{67}{132} < \theta_i \le 1$. As a result, candidate C wins the plurality election, and in the proportional stage L and R take, respectively, $\frac{31}{66}$ and $\frac{35}{66}$ of the votes.

4.2 Representative democracy

In this section we analyze what can happen when the set of candidates is not exogenous. At this end, we adopt a model analogous to Besley and Coate (1997). We consider a community consisting of a set of citizens N that, in order to implement a policy X, must elect some representatives among themselves.

The selection of the community representatives requires an election. Each citizen is allowed to run for election, acting as a candidate. All citizens choosing to be a candidate face a utility cost δ.

The political process consists of a three-stage game. In the first stage, each citizen decides whether to become a candidate or not. In the second stage, the election occurs. In the third stage, the policy is implemented. In Besley and Coate's (1997) model the election is run with plurality rule and, because there is no commitment, each elected candidate implements his preferred policy.

Let us consider what happens when the election is run with proportional rule and the policy is given by:

$$X = \sum_{k=1}^{m} \zeta_k v_k$$

where v_k denotes the share of votes taken by citizen-candidate k.[21]

If we let the number of citizens go to infinity, we know that for a given set of candidates only the two extremists will take votes. Hence in every pure strategy subgame perfect equilibrium we will have only two candidates. Moreover, assuming $\tilde{\theta} \notin \{0, 1\}$, we have:

$$\frac{\partial \tilde{\theta}}{\partial \zeta_L} = \frac{H(\tilde{\theta})}{1 + (\zeta_R - \zeta_L)H'(\tilde{\theta})} > 0$$
$$\frac{\partial \tilde{\theta}}{\partial \zeta_R} = \frac{1 - H(\tilde{\theta})}{1 + (\zeta_R - \zeta_L)H'(\tilde{\theta})} > 0.$$

This implies that a more extreme citizen, if he decides to be a candidate, will move the outcome toward him. Hence, for a given cost of candidacy, if the leftmost candidate is sufficiently far from the extremist citizen, the latter will prefer to become a candidate. As a result, in every pure strategy equilibrium, as the cost of candidacy goes to zero, only the two extremists decide to become candidates.

Remark 3 Policy oriented parties: The fact that $\frac{\partial \tilde{\theta}}{\partial \zeta_L} > 0$ and $\frac{\partial \tilde{\theta}}{\partial \zeta_R} > 0$ has an interesting implication in a model where there are two policy-oriented parties that can commit to a policy before the election is called. The equilibrium choices

[21] To avoid confusion we still denote ζ_i as the preferred policy of candidate i. We have proved the basic results for a finite number of parties, hence we have to assume that the number of candidates is finite. This is not an assumption when we consider the game with a continuum of citizens as an "approximation" of the game with a finite number of players.

of the parties do not converge toward centrist policy, but either both parties are "radical" (i.e., the policies they commit to will be respectively 0 and 1) or one is "radical" and the outcome coincides with the preferred policy of the other, the choice of the latter being, however, more extremist than its preferred policy. A similar result has been proved with sincere voting and further assumptions on the distribution of voters by Ortuño-Ortin (1997), while we obtain it with strategic voting.[22] Furthermore, Alesina and Rosenthal (2000) prove, under incomplete information, that parties offer divergent platforms, when they care both about winning and about the policy, which is a compromise between the executive and the legislature.

5 Conclusion

In this paper we have analyzed strategic voting in proportional rule elections, in the attempt to cover a lack in the literature of strategic voting in mass elections. As a matter of fact, scholars mostly focused on plurality rule elections, while proportional rule received little attention. The main explanation is indeed the determination of the policy outcome. In plurality elections, the policy outcome is unambiguously the preferred one of the winner, i.e. the winner-takes-all story. But if we want to study proportional representation, the definition of the policy outcome is much more controversial. Clearly, one should consider the outcome as the result of a bargaining game among elected politicians. In this paper we deliberately have chosen to focus on strategic behavior of individual voters, who face a policy outcome that is simply a weighted average of parties' platforms with weights equal to the share of votes. Nonetheless, we believe that this policy outcome captures the spirit of proportional rule: all parties represented in parliament contribute, with powers equal to their share of votes, to the policy formation. Furthermore, as we pointed out, such an outcome function has a nice justification, being the utilitarian solution of a bargaining game among politicians with a quadratic loss function.

The main contribution of this paper, compared to the existing literature, is to solve a very simple game, in which a finite number of voters may vote for any party belonging to a finite set M. On top of that, we do not impose any particular restrictions neither on voters' preferences, except the single-peakedness, neither on the distribution of players' bliss points. The main intuition of the paper is quite "obvious": under proportional representation strategic voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their ideal point.

Acknowledgements We would like to thank Tim Besly, Leo Ferraris, Michel Le Breton, Jean-François Mertens, Thomas Palfrey, Howard Rosenthal, Guido Tabellini, and Jacques Thisse for

[22] A move toward a more extreme position produces two effects: on one hand the number of votes decreases, on the other hand the votes are on a more extreme position. With sincere voting the net effect can be either positive or negative, depending upon the distribution of the voters, whereas with strategic voting the second effect always dominates the first one.

helpful discussions and remarks. We also thank an anonymous referee for comments. This project has been carried over for several years, and the authors wish to thank the institutions that gave them hospitality during this period: CORE, the Wallis Institute of Political Economy, Tor Vergata University, STICERD (Francesco) and University of Cambridge (Giovanna).

Appendix

Proof of Proposition 3
Given a mixed strategy σ_j, the player j's vote is a random vector[23] $\tilde{s}_j$ with $Pr\left(\tilde{s}_j = k\right) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, \ldots \sigma_{i-1}, \sigma_{i+1}, \ldots \sigma_n)$, let $\bar{\tilde{s}}^{-i} = \frac{1}{n-1} \sum_{j \in N/\{i\}} \tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1} \sum_{j \in N/i} \sigma_j$. The first step of the proof consists in proving the following lemma:

Lemma 5 $\forall \phi > 0$ *and* $\forall \delta > 0$, *if* $n > \frac{m}{4\phi^2\delta} + 1$, *then* $\forall \sigma, \forall i$

$$Pr\left(\mid \bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}} \mid \leq \vec{\phi} \right) > 1 - \delta.$$

Proof To prove the lemma we can use Chebychev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $\text{Var}(\tilde{s}_j^k) = \sigma_j^k(1 - \sigma_j^k) \leq \frac{1}{4}$, hence $E(\bar{\tilde{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and $\text{Var}(\bar{\tilde{s}}_k^{-i}) \leq \frac{1}{4(n-1)}$. By Chebychev's inequality we know that $\forall k, \forall \phi$:

$$Pr\left(\mid \bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}} \mid > \phi \right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$Pr\left(\mid \bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}} \mid \leq \vec{\phi} \right) \geq 1 - \sum_k Pr\left(\mid \bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}} \mid > \phi \right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1 - \delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. □

Lemma 6 $\forall \varepsilon > 0$, $\exists n_0^{\mathrm{L}}$ *such that* $\forall n \geq n_0^{\mathrm{L}}$, *if the game has n voters and if* $\theta_i \leq X\left(\bar{\mu}^{\sigma}\right) - \varepsilon$, *then* L *is the only best reply for player i to* σ^{-i}.

Proof Fix $\varepsilon > 0$. Define $\forall \theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]$

$$M_{\varepsilon}\left(\theta\right) = \max_{X \in \left[\theta + \frac{\varepsilon}{2}, 1\right]} \frac{\partial u(X, \theta)}{\partial X}.$$

[23] We remind readers that a vote is a vector with m components. Thereafter, given a scalar α, we denote with $\vec{\alpha}$ the vector with m components, all of them equal to α, while given a vector $\beta = (\beta_1, \ldots, \beta_m)$ with $\mid \beta \mid$ we denote the vector $\left(\mid \beta_1 \mid, \ldots, \mid \beta_m \mid\right)$.

By single-peakedness we know that $M_\varepsilon(\theta) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ we can apply the theorem of the maximum[24] to deduce that the function $M_\varepsilon(\theta)$ is continuous, hence it has a maximum on $\left[0, 1-\frac{\varepsilon}{2}\right]$, which is strictly negative. Let

$$M_\varepsilon^* = \max_{\theta \in \left[0,1-\frac{\varepsilon}{2}\right]} M_\varepsilon(\theta).$$

Let M denote the upper bound[25] of $\left| \frac{\partial u(X,\theta)}{\partial X} \right|$ on $[0,1]^2$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*}{M - M_\varepsilon^*} > 0$ and $\phi^* = \frac{\left(-2+\sqrt{6}\right)\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, then every strategy other than L cannot be a best reply for player i, which, setting n_0^{L} equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, directly implies the claim.

Take a party $\mathrm{c} \neq \mathrm{L}$. By definition $\mathrm{c} \in \mathrm{BR}_i(\sigma) \Longrightarrow$

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, \mathrm{c}), \theta_i\right) - u\left(X(s_{-i}, \mathrm{L}), \theta_i\right) \right] \geq 0, \tag{4}$$

which can be written as:

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, \mathrm{c}), \theta_i\right) - u\left(X(s_{-i}, \mathrm{c}) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i \right) \right] \geq 0. \tag{5}$$

Because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, (5) can be written as:

$$\sum_{v_n^{-i} \in V_n^{-i}} Pr\left(\bar{\tilde{s}}^{-i} = v_n^{-i} \right) \left[u\left(X\left(v_n^{-i}, \mathrm{c}\right), \theta_i\right) - u\left(X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i \right) \right] \geq 0, \tag{6}$$

where with abuse of notation, $X\left(v_n^{-i}, \mathrm{c}\right) = \frac{\zeta_{\mathrm{c}}}{n} + \frac{n-1}{n} \sum_{k=1}^{m} \zeta_k v_{n(k)}^{-i}$. Multiplying both sides of (6) by $\frac{n}{\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}} > 0$ we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} Pr\left(\bar{\tilde{s}}^{-i} = v_n^{-i} \right) \frac{\left[u\left(X\left(v_n^{-i}, \mathrm{c}\right), \theta_i\right) - u\left(X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i \right) \right]}{\frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}})} \geq 0. \tag{7}$$

[24] Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using (cf. Theorem 3.6 in Stokey and Lucas 1989). Let $f : \Psi \times \Phi \to \Re$ be a continuous function and $g : \Phi \to P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\{f(\psi, \phi) \mid \psi \in g(\phi)\}$ is continuous on Φ.

[25] The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

By the *mean value theorem* we know that $\forall v_n^{-i}$,
$\exists X^* \in \left[X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), X\left(v_n^{-i}, \mathrm{c}\right)\right]$ such that

$$\frac{\left[u\left(X\left(v_n^{-i}, \mathrm{c}\right), \theta_i\right) - u\left(X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i\right)\right]}{\frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}})} = \left.\frac{\partial u(X, \theta_i)}{\partial X}\right|_{X=X^*}.$$

Let us define now the upper bound of $\frac{\partial u(X,\theta_i)}{\partial X}$ for $X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, \mathrm{c}) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), 1\right]$:

$$M_n^*(\vec{\phi}^*, \theta_i) = \max_{X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, \mathrm{c}) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), 1\right]} \frac{\partial u(X, \theta_i)}{\partial X}.$$

Recalling that M is the upper bound of $\mid \frac{\partial u(X,\theta)}{\partial X} \mid$ on $[0,1]^2$ the following inequality follows:

$$\sum_{v_n^{-i} \in V_n^{-i}} Pr\left(\bar{\tilde{s}}^{-i} = v_n^{-i}\right) \frac{\left[u\left(X\left(v_n^{-i}, \mathrm{c}\right), \theta_i\right) - u\left(X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i\right)\right]}{\frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}})}$$
$$\leq Pr\left(\mid \bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}} \mid \leq \vec{\phi}^*\right) M_n^*(\vec{\phi}^*, \theta_i) + \left(1 - \Pr\left(\mid \bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}} \mid \leq \vec{\phi}^*\right)\right) M.$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon^*$. From the definition of M_ε^*, it suffices to prove that $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon(\theta_i)$, which is true if $X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, \mathrm{c}) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}})$ is greater than $\theta_i + \frac{\varepsilon}{2}$.

Therefore, we have:

$$\begin{aligned}
&X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, \mathrm{c}) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}) \\
&= \frac{n-1}{n} \sum_k \bar{\mu}_k^{\sigma_{-i}} \zeta_k - \frac{n-1}{n} \sum_k \phi^* \zeta_k + \frac{1}{n} \zeta_{\mathrm{L}} \\
&= X(\bar{\mu}^{\sigma}) - \frac{1}{n} \sum_k \sigma_i^k \zeta_k + \frac{1}{n} \zeta_L - \frac{n-1}{n} \sum_k \phi^* \zeta_k \\
&> X(\bar{\mu}^{\sigma}) - \frac{1}{n}(\zeta_{\mathrm{R}} - \zeta_{\mathrm{L}}) - m\phi^* \zeta_{\mathrm{R}} \geq \theta_i + \varepsilon - \frac{1}{n} - m\phi^*.
\end{aligned}$$

Hence this step of the proof is concluded by noticing that δ^*_ε is by definition less than $\frac{1}{2}$, hence [26]

$$\begin{aligned}
\theta_i + \varepsilon - \frac{1}{n} - m\phi^* &> \theta_i + \varepsilon - \frac{2\phi^{*2}}{m} - m\phi^* \\
&= \theta_i + \varepsilon - \frac{(20-8\sqrt{6})\varepsilon^2}{m^3} - \varepsilon\left(-2+\sqrt{6}\right) \geq \theta_i \\
&\quad + \varepsilon\left(1 - \frac{(20-8\sqrt{6})}{8} + 2 - \sqrt{6}\right) \\
&= \theta_i + \frac{1}{2}\varepsilon.
\end{aligned}$$

By Lemma 5 and the above claim, we know that, for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon} + 1$,

$$\begin{aligned}
&\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right) M^*_n(\vec{\phi}^*, \theta_i) + \left(1 - \Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right)\right) M \\
&< (1-\delta^*_\varepsilon) M^*_\varepsilon + \delta^*_\varepsilon M = \left(1 - \frac{-M^*_\varepsilon}{M - M^*_\varepsilon}\right) M^*_\varepsilon + \frac{-M^*_\varepsilon}{M - M^*_\varepsilon} M = 0.
\end{aligned}$$

Summarizing, we have proved that for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon} + 1$, for every strategy $\mathrm{c} \neq \mathrm{L}$

$$\begin{aligned}
&\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, \mathrm{c}\right), \theta_i\right) - u\left(X\left(v_n^{-i}, \mathrm{c}\right) - \frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}}), \theta_i\right)\right]}{\frac{1}{n}(\zeta_{\mathrm{c}} - \zeta_{\mathrm{L}})} \\
&\leq \Pr\left(\mid \bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}} \mid \leq \vec{\phi}^*\right) M^*_n(\vec{\phi}^*, \theta_i) + \left(1 - \Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right)\right) M \\
&< (1-\delta^*_\varepsilon) M^*_\varepsilon + \delta^*_\varepsilon M = 0,
\end{aligned}$$

which implies that c is not a best reply for player i. □

Analogously, it can be proved the following Lemma.

Lemma 7 *$\forall \varepsilon > 0$, $\exists n_0^{\mathrm{R}}$ such that $\forall n \geq n_0^{\mathrm{R}}$, if the game has n voters and if $\theta_i \geq X(\bar{\mu}^\sigma) + \varepsilon$, then R is the only best reply for player i to σ^{-i}.*

Setting $n_0 = \max\left\{n_0^{\mathrm{L}}, n_0^{\mathrm{R}}\right\}$ completes the proof.

References

Alesina A, Rosenthal H (1995) Partisan politics, divided government, and the economy. Cambridge University Press, Cambridge

[26] In the following we assume that $\varepsilon \leq 1$, since otherwise the proposition is trivially true.

Alesina A, Rosenthal H (1996) A theory of divided government. Econometrica 64:1311–1341

Alesina A, Rosenthal H (2000) Polarized platforms and moderate policies with checks and balances. J Public Econ 75:1–20

Austen-Smith D (2000) Redistributing income under proportional representation. J Polit Econ 108:1235–1269

Austen-Smith D, Banks J (1988) Elections, coalitions, and legislative outcomes. Am Polit Sci Rev 82:405–422

Baron DP, Diermeier D (2001) Elections, governments, and parliaments in proportional representation systems. Q J Econ 116:933–967

Besley T, Coate S (1997) An economic model of representative democracy. Q J Econ 112:85–114

Carty RK (1988) Ireland: from predominance to competition. In: Wolinetz S (ed) Parties and party systems in liberal democracies. Routledge, London

Cox G (1994) Strategic voting equilibria under the single nontransferable vote. Am Polit Sci Rev 88:608–621

Cox G (1997) Making votes count. Cambridge University Press, Cambridge

De Sinopoli F (2000) Sophisticated voting and equilibrium refinements under plurality rule. Soc Choice Welf 17:655–672

Duverger M (1954) Political parties. Methuen, London

Engelmann FC (1988) The austrian party system: continuity and change. In: Wolinetz S (eds.) Parties and party systems in liberal democracies. Routledge, London

Fey M (1997) Stability and coordination in Duverger's law: a formal model of preelection polls and strategic voting. Am Polit Sci Rev 91:135–147

Grossman G, Helpman E (1999) Competing for endorsment. Am Econ Rev 89:501–524

Hotelling H (1929) Stability in competition. Econ J 39:41–57

Iannantuoni G (2004) A purely non-cooperative model of divided government. Soc Choice Welf 22:401–412

Leys C (1959) Models, theories and the theory of political parties. Polit Stud 7:127–146

Morelli M (2004) Party formation and policy outcomes under different electoral systems. Rev Econ Stud 71:829–853

Myerson RB, Weber RJ (1993) A Theory of voting equilibria. Am Polit Sci Rev 87:102–114

Ortuño-Ortin I (1997) A spatial model of political competition and proportional representation. Soc Choice Welf 14:427–438

Palfrey T (1989) A mathematical proof of Duverger's law. In: Ordeshook PC (ed) Models of strategic choice in politics. University of Michigan Press, Ann Arbor, pp 69–91

Riker WH (1982) The two-party system and Duverger's law: an essay on the history of political science. Am Polit Sci Rev 76:753–766

Sartori G (1968) Political development and political engineering. In: Montgomery JD, Hirschman AO (eds) Public policy, vol. XVII. Harbard University Press, Cambridge, pp. 261–298

Shugart MS (1995) The electoral cycle and institutional sources of divided presidential government. Am Polit Sci Rev 89:327–343

Stokey NL, Lucas RE (1989) Recursive methods in economic dynamics. Harvard University Press, Cambridge